Topological Quantum Field Theories

Thomas Wasserman, University of Oxford

November 19, 2014

Origin of TQFT

- Manifold invariants [Donaldson, Floer, around '83] from gauge field theory
- Quantum field theory based knot invariants conjectured by physicists [Witten, '88]
- Made rigorous, called Topological Quantum Field Theory [Atiyah, Turaev, Reshetikhin, around '91]

Basic definition

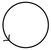
Definition (Atiyah)

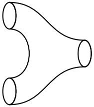
Let k be a field. A *topological quantum field theory* of dimension n is a monoidal functor $Z : \mathbf{Bord}^{\mathrm{or}}_{n-1,n} \to \mathbf{Vect}(k)$.

Slogan: a TQFT is a representation of the bordism category.

Bordism Categories

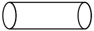
Let n be a positive integer. Define the category $\mathbf{Bord}^{\text{or}}_{n,n+1}$ as follows:

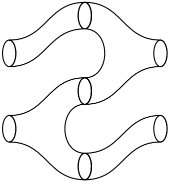
- Objects: Closed oriented $(n-1)$-manifolds, M. 
- Morphisms (maps): Bordisms M to N: that is, an oriented n-dimensional manifold B equipped with an orientation-preserving diffeomorphism $\partial B \simeq \bar{M} \coprod N$.



$B = B'$ if there is an orientation-preserving diffeomorphism $B \simeq B'$ extending $\partial B \simeq \bar{M} \coprod N \simeq \partial B'$.

Categorical structure

- For $M \in \mathbf{Bord}^{\mathrm{or}}_{n,n+1}$, id_M is the product bordism $B = M \times [0,1]$.


- Composition of maps is by gluing of bordisms: For $M, M', M'' \in \mathbf{Bord}^{\mathrm{or}}_{n,n+1}$, $B : M \to M'$, $B' : M' \to M''$, $B' \circ B$ is represented by the manifold $B \coprod_{M'} B'$.


- $\mathbf{Bord}^{\mathrm{or}}_{n,n+1}$ has a *monoidal structure*, $\otimes$, by taking $M \otimes M' := M \sqcup M'$.

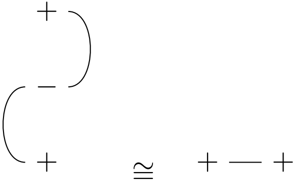
Dimension 1

- Objects are oriented points=0-manifolds: $+$ $-$
- $Z(+) = V$, $Z(-) = W$.
- Maps are:

id_V or id_W	eval	coeval	eval $\circ$ coeval
	$V \otimes W \to k$	$k \to W \otimes V$	$k \to k$
	$(v, \lambda) \mapsto \lambda(v)$	$x \mapsto x\,\mathrm{id}_V$	$\mathrm{tr}(\mathrm{id}_V) = \dim V$

Relations

- Observe that:



- Together with the same for $-$, this gives $W = V^*$.
- 1d TQFTs $\leftrightarrow$ finite dimensional vector spaces.

Generalising the definition

- Structure: orientations can be replaced with eg: Spin, Framed, Euclicidean
- Extension: consider $(n-1)$-bordisms between $(n-2)$-manifolds and n-bordism between those
- Target: replace **Vect** by your favourite monoidal (higher) category

For me: Spin, $(1,2,3)$, **LinCat**.

Modern motivation

- (Fully extended) TQFTs are intimately related to higher categories [Baez, Dolan, Lurie, Teleman, Kapustin, ...]
- Visualising and studying complicated algebraic structures [Bartlett, Douglas, Schommer-Pries, Vicary, ...]
- Applications to (condensed matter) physics [Walker, Wang, Kitaev, Freed, Freedman, ...]
- Topological Modular Forms (Elliptic cohomology/"Higher K-theory") [Hopkins, Stolz, Teichner, Douglas, Henriques, ...]
- Geometric Langlands [Beilinson, Drinfeld, Kapustin, Witten, Frenkel, Gaitsgory, ...]

Faculty working on related topics

Topology

- Chris Douglas (my supervisor)
- Andras Juhasz
- Constantin Teleman
- Ulrike Tillmann

Algebra

- Kobi Kremitzer
- Kevin McGerty

Geometry

- Nigel Hitchin
- Alexander Ritter